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Vibrations of angle-ply laminated circular cylindrical shells subjected to different sets of edge boundary conditions

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Abstract. This paper studies the free vibrations of finite, closed, circular cylindrical shells, made of one or more monoclinic layers. The study is based on the Love-type version of a unified shear-deformable shell theory. This theory enables the trial and testing of different through-thickness transverse shear-strain distributions and, among them, strain distributions that do not involve the undesirable implications of the transverse-shear correction factors. For flexural vibrations, the analytical solution of the corresponding axisymmetric solution is obtained, as a particular case, when it is assumed that the free-vibration pattern is independent of the circumferential co-ordinate parameter. If the appropriate material simplifications are employed, the present analysis yields, as a further particular case, the corresponding free-vibration solution that has already been presented elsewhere for cross-ply laminated cylindrical shells.

Key words: vibrations, continuous systems, laminates, composites, shells, boundary conditions, interlaminar stresses.

1. Introduction

The subject of the mechanics of composite materials and structures has experienced tremendous growth in the last three decades. Due to the increasing use of light-weight, high-strength and high-stiffness materials in the aerospace industry, as well as in mechanical and marine applications, the study and the understanding of the behaviour of composite structural elements subjected to static, dynamic or thermal loading is of profound importance. With the fast development of powerful computers and relevant computer codes, the analyst is given today an opportunity to achieve this by using advanced mathematical modelling and applied-mathematics methods to an extent that was unknown a few decades ago. Difficult problems concerning, for instance, the static or the dynamic behaviour of shell-type structural elements made of highly reinforced layered materials can now be tackled with a relative ease, by means of old or new mathematical models and methods of variable difficulty.

Composite cylindrical shells are among the most frequently used structural elements. As a consequence, the existing literature on the dynamic analysis of circular cylindrical shells and open panels is impressively extensive (see, for instance, references [1, pp. 185–218], [2–5]). The vast majority of the relevant analytical studies dealt, however, with thin or moderately thick cylindrical shells, the material anisotropy of which is no more complicated than that of the special orthotropy (material axes of orthotropy coincide with the axes of the orthogonal curvilinear co-ordinate system [6, pp. 31–59]).

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As far as the free vibrations of closed and finite circular cylindrical shells are concerned, the current developments can be considered satisfactory only for shells made of one or more isotropic, transversely isotropic or specially orthotropic layers; namely, up to the material arrangement of a cross-ply laminate. The latest relevant developments [7–11], [12, pp. 48–51] make further clear that, regardless of the classical or the shear deformable shell theory employed, the state space concept can provide an analytical solution to such problems for any set of boundary conditions imposed on the two shell edges. These studies [7–11], [12, pp. 48–51], as well as numerous others that can be found through them, contain an extensive number of particular example applications and discuss a considerable amount of associated numerical results. On the contrary, there is comparatively little done [13–17] for the understanding of the dynamic behaviour of closed cylindrical shells made of one or more generally anisotropic, monoclinic or even generally orthotropic layers (angle-ply laminates).

Sun and Whitney [13] and Soldatos and Ye [14] studied the axisymmetric vibrations of finite composite cylindrical shells made of generally orthotropic or monoclinic layers. In [13], which appears to be the first study in the subject, the model employed was based on an advanced two-dimensional shell theory that takes both transverse shear and transverse normal deformation effects into consideration. As is well documented and widely recognised, consideration of the transverse deformation effects (particularly those accounting for transverse shear) is very important when dealing with vibrations of highly reinforced composite structural elements. The higher-order effects involved in [13] were modelled, however, in a rather unrealistic manner. Hence, the resulted shell theory made use of an excessive number of shear correction factors (six), the accurate determination of which is not possible. The solution of the axisymmetric vibration problem considered in [13] was obtained by the represention of the unknown displacement functions in a relatively simple trigonometric from, which satisfies a certain type of simply supported edge boundary conditions.

Most recently, Soldatos and Ye [14] considered and studied the same essentially axisymmetric vibration problem, together with its static equivalent. The model employed in [14] was based, however, on entirely three-dimensional equations of elasticity, a most accurate solution of which was obtained by the use of a successive approximation method. This solution was obtained for a set of edge boundary conditions that can essentially be regarded as the point-by-point equivalent of the simply supported edge conditions considered in [13].

Vanderpool and Bert [15] studied the flexural (non-axisymmetric) free vibrations of closed, homogeneous (single-layered) monoclinic cylindrical shells, the edges of which are subjected to different sets of boundary conditions. Their model was based on a uniform shear-deformable shell theory, which makes use of three shear-correction factors. For the solution of the corresponding differential equations of motion they used a so-called semi-inverse analytical method suggested by Flügge [18, pp. 222–225] and demonstrated by Forsberg [19] for thin isotropic cylindrical shells. It should be noted that this method is essentially equivalent with employing the state space concept [7–12] on the same set of differential equations. Hence, it is applicable regardless of the particular set of edge boundary conditions involved. The solution obtained in [15] was used for the derivation of relatively few natural frequencies of homogeneous monoclinic shells having both their edges free of external tractions. These were, however, used for a comparison with corresponding natural frequencies that were identified experimentally.

For the flexural vibration problem of closed angle-ply laminated cylindrical shells, Soldatos [16] made use of the so-called parabolic shear-deformable shell theory that avoids the undesirable implications of the transverse shear-correction factors. Both shell edges were assumed as being subjected to a certain type of simple supports. For that type of edge boundary conditions, the solution of the differential equations of motion was obtained upon application of Galerkins's method. It should be further mentioned that the so-called helical modal pattern approach [20, 21] was also used in [16]. This leads to an exact closed-form solution of the equations of motion, which, however, cannot satisfy any set of edge boundary conditions. It can therefore be used only for the study of the propagation of harmonic waves in angle-ply laminated cylinders of infinite extent.

Narita *et al.* [17] studied the flexural (non-axisymmetric) free vibrations of closed, laminated composite cylindrical shells made of different monoclinic layers. Although the model was only based on a second approximation classical shell theory of the Flügge-type [18], the vibration solution achieved in [17] is applicable regardless of the particular set of the edge boundary conditions employed. In more detail, all three unknown displacement functions are expressed in appropriate power series of the middle-surface co-ordinate parameters and the unknown constant coefficients involved are determined upon application of the Ritz method on the corresponding energy functional. Moreover, a substantial amount of numerical results was presented and discussed in [17], which, from that point of view, appears to be the most complete study that has appeared so far in its subject.

This paper studies both the axisymmetric and the flexural free vibrations of finite, closed, circular cylindrical shells, made of one or more monoclinical layers. The study is based on the Love-type version of the unified shear-deformable shell theory presented in [22]. This theory enables the trial and testing of different through-thickness transverse shear-strain distributions and, among them, strain distributions that do not involve the undesirable implications of the transverse sher correction factors. For flexural vibrations, the analytical solution of the corresponding equations of motion is obtained on the basis of the state space concept [7–12]. Hence, it is applicable regardless of the boundary conditions imposed on the shell edges. The corresponding axisymmetric solution can then be obtained, as a particular case, when it is assumed that the free vibration pattern is independent of the circumferential co-ordinate parameter.

2. Theory

Adopting the usual notation, we denote with L, R and h the length, the middle-surface radius and the thickness, respectively, of the closed cylindrical shell considered. The axial, the circumferential and the transverse to the middle-surface co-ordinate parameters are denoted with x, s and z, respectively, and are such that,

$$0 \leqslant x \leqslant L, \qquad 0 \leqslant s \leqslant 2\pi R, \qquad -h/2 \leqslant z \leqslant h/2. \tag{1}$$

The corresponding displacement components are denoted with U, V and W and are functions of the spatial co-ordinates and the time, t. The shell is assumed as made of an arbitrary number, say N, of monoclinic elastic layers. Hence, the stress state in its kth layer, counting from the inner to the outer layer, is governed by the following form of the generalised Hooke's law ($k = 1, 2, \ldots, N$):

$$\begin{bmatrix} \sigma_x^{(k)} \\ \sigma_s^{(k)} \\ \tau_{xs}^{(k)} \end{bmatrix} = \begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} & Q_{16}^{(k)} \\ Q_{12}^{(k)} & Q_{22}^{(k)} & Q_{26}^{(k)} \\ Q_{16}^{(k)} & Q_{26}^{(k)} & Q_{66}^{(k)} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_s \\ \gamma_{xs} \end{bmatrix}, \begin{bmatrix} \tau_{sz}^{(k)} \\ \tau_{xz}^{(k)} \end{bmatrix} = \begin{bmatrix} Q_{44}^{(k)} & Q_{45}^{(k)} \\ Q_{45}^{(k)} & Q_{55}^{(k)} \end{bmatrix} \begin{bmatrix} \gamma_{sz} \\ \gamma_{xz} \end{bmatrix}, \tag{2}$$

where $Q_{ij}^{(k)}$ are the well-known reduced elastic stiffnesses [6]. It should be noted that the material arrangement of a cross-ply laminated cylinder is described with a particular form of Equation (2), namely $Q_{16}^{(k)} = Q_{26}^{(k)} = Q_{45}^{(k)} = 0$. It should be further made clear that the material arrangement of a so-called angle-ply laminate (a laminate made of generally orthotropic layers [6]) is also a particular case of the arrangement described by (2). In what follows, however, it is convenient to occasionally associate with (2) the term 'angle-ply laminate', despite that Equation (2) represents a more general material arrangement.

The formulation of the unified shear deformable shell theory presented in [22] starts with the following displacement approximation:

$$\begin{aligned} U(x,s,z;t) &= u(x,s;t) - zw_{,x} + \Phi_1(z)u_1(x,s;t), \\ V(x,s,z;t) &= (1+z/R)v(x,s;t) - zw_{,s} + \Phi_2(z)v_1(x,s;t), \\ W(x,s,z;t) &= w(x,s;t). \end{aligned} \tag{3}$$

Here u, v, w, u_1 and v_1 are the five unknown displacement functions (degrees of freedom), while the shape functions $\Phi_1(z)$ and $\Phi_2(z)$ are to be specified *a posteriori*. For Love-type shell approximations, the displacement model (3) yields the following nonzero strain components:

$$\begin{aligned} \varepsilon_x &= u_{,x} - zw_{,xx} + \Phi_1(z)u_{1,x}, \\ \varepsilon_s &= (1+z/R)v_{,s} - zw_{,ss} + \Phi_2(z)v_{1,s} + w/R, \\ \gamma_{sz} &= \Phi_2' v_1, \\ \gamma_{xz} &= \Phi_1' u_1, \\ \gamma_{xs} &= u_{,s} + v_{,x} + z(-2w_{,xs} + v_{,x}/R) + \Phi_1 u_{1,s} + \Phi_2 v_{1,x}, \end{aligned} \tag{4}$$

where a prime denotes ordinary differentiation with respect to z. It becomes therefore clear that, through their derivatives, the *a posteriori* specified functions $\Phi_1(z)$ and $\Phi_2(z)$ will determine the through-the-thickness trial distribution of the transverse shear strains.

The force and moment resultants of the theory are defined according to,

$$\begin{aligned} (N_x, N_s, N_{xs}) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_s, \tau_{xs})\mathrm{d}z, & (M_x, M_s, M_{xs}) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_s, \tau_{xs})z\,\mathrm{d}z, \\ (M_x^a, M_{xs}^a) &= \int_{-h/2}^{h/2} (\sigma_x, \tau_{xs})\Phi_1(z)\mathrm{d}z, & (M_s^a, M_{sx}^a) &= \int_{-h/2}^{h/2} (\sigma_s, \tau_{sx})\Phi_2(z)\mathrm{d}z, \\ Q_x^a &= \int_{-h/2}^{h/2} \tau_{xz}\Phi_1'\mathrm{d}z, & Q_s^a &= \int_{-h/2}^{h/2} \tau_{sz}\Phi_2'\,\mathrm{d}z, \end{aligned} \tag{5}$$

and, after Equations (2), they yield the following constitutive equations:

$$\begin{bmatrix} N_x \\ N_s \\ N_{xs} \\ M_x \\ M_s \\ M_{xs} \\ M_x^a \\ M_s^a \\ M_{xs}^a \\ M_{sx}^a \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & B_{111} & B_{122} & B_{161} & B_{162} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & B_{121} & B_{222} & B_{261} & B_{262} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & B_{161} & B_{262} & B_{661} & B_{662} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & D_{111} & D_{122} & D_{161} & D_{162} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & D_{121} & D_{222} & D_{261} & D_{262} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & D_{161} & D_{262} & D_{661} & D_{662} \\ B_{111} & B_{121} & B_{161} & D_{111} & D_{121} & D_{161} & D_{1111} & D_{1212} & D_{1611} & D_{1612} \\ B_{122} & B_{222} & B_{262} & D_{122} & D_{222} & D_{262} & D_{1212} & D_{2222} & D_{2612} & D_{2622} \\ B_{161} & B_{261} & B_{661} & D_{161} & D_{261} & D_{661} & D_{1611} & D_{2612} & D_{6611} & D_{6612} \\ B_{162} & B_{262} & B_{662} & D_{162} & D_{262} & D_{662} & D_{1612} & D_{2622} & D_{6612} & D_{6622} \end{bmatrix}$$

$$\times \begin{bmatrix} u_{,x} \\ v_{,s} + \dfrac{w}{R} \\ u_{,s} + v_{,x} \\ -w_{,xx} \\ -w_{,ss} + \dfrac{v_{,s}}{R} \\ -2w_{,xs} + \dfrac{v_{,x}}{R} \\ u_{1,x} \\ v_{1,s} \\ u_{1,s} \\ v_{1,x} \end{bmatrix}, \begin{bmatrix} Q_s^a \\ Q_x^a \end{bmatrix} = \begin{bmatrix} A_{4422} & A_{4512} \\ A_{4512} & A_{5511} \end{bmatrix} \begin{bmatrix} v_1 \\ u_1 \end{bmatrix}. \tag{6}$$

Here, the appearing rigidities are defined as follows:

$$\begin{aligned} A_{ij} &= \int_{-h/2}^{h/2} Q_{ij}^{(k)}\, \mathrm{d}z, \quad A_{ijlm} = \int_{-h/2}^{h/2} Q_{ij}^{(k)} \Phi_l' \Phi_m'\, \mathrm{d}z, \\ B_{ij} &= \int_{-h/2}^{h/2} Q_{ij}^{(k)} z\, \mathrm{d}z, \quad B_{ijl} = \int_{-h/2}^{h/2} Q_{ij}^{(k)} \Phi_l\, \mathrm{d}z, \\ D_{ij} &= \int_{-h/2}^{h/2} Q_{ij}^{(k)} z^2\, \mathrm{d}z, \quad D_{ijl} = \int_{-h/2}^{h/2} Q_{ij}^{(k)} \Phi_l z\, \mathrm{d}z, \quad D_{ijlm} = \int_{-h/2}^{h/2} Q_{ij}^{(k)} \Phi_l \Phi_m\, \mathrm{d}z, \end{aligned} \tag{7}$$

with their indices taking appropriate integer values.

The five variationally consistent equations of motion of the theory, given in terms of the force and moment resultants (5), are as follows [11, 22]:

$$N_{x,x} + N_{xs,s} = I_1,$$

$$
\begin{aligned}
&N_{xs,x} + N_{s,s} + \frac{1}{R}(M_{s,s} + M_{xs,x}) = I_2,\\
&-\frac{1}{R}N_s + M_{x,xx} + (M_{xs} + M_{sx})_{,xs} + M_{s,ss} = I_3,\\
&M^a_{x,x} + M^a_{xs,s} - Q^a_x = I_4,\\
&M^a_{sx,x} + M^a_{s,s} - Q^a_s = I_5,
\end{aligned}
\tag{8}
$$

where the appearing terms are defined according to,

$$
\begin{aligned}
&I_1 = (\rho_0 u - \rho_1 w_{,x} + \overline{\rho}^{11}_0 u_1)_{,tt},\\
&I_2 = \{[\rho_0 + (2\rho_1 + \rho_2/R)/R]v + (\rho_1 - \rho_2/R)w_{,s} + (\overline{\rho}^{21}_0 + \overline{\rho}^{21}_1/R)v_1\}_{,tt},\\
&I_3 = \{\rho_0 w - (\rho_1 - \rho_2/R)v_{,s} - \rho_2(w_{,xx} + w_{,ss}) + \rho_1 u_{,x} + \overline{\rho}^{11}_1 u_{1,x} + \overline{\rho}^{21}_1 \overline{v}_{1,s}\}_{,tt},\\
&I_4 = (\overline{\rho}^{11}_0 u - \overline{\rho}^{21}_1 w_{,x} + \rho^{12}_0 u_1)_{,tt},\\
&I_5 = \{(\overline{\rho}^{21}_0 + \overline{\rho}^{21}_1/R)v - \overline{\rho}^{21}_1 w_{,s} + \overline{\rho}^{22}_0 v_1\}_{,tt},
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
\rho_i &= \int_{-h/2}^{h/2} \rho z^i \,\mathrm{d}z, \quad (i = 0, 1, 2)\\
\overline{\rho}^{lm}_i &= \int_{-h/2}^{h/2} \rho z^i \Phi^m_l \,\mathrm{d}z, \quad (i = 0, 1; l, m = 1, 2).
\end{aligned}
\tag{10}
$$

Upon inserting Equations (6) and (9) into the equations of motion (8), one finally obtains five Navier-type partial differential equations of motion, in terms of the five main unknown displacement functions. These can be represented in the following differential eigenvalue form:

$$
[L]\{\delta\} = \{0\}, \{\delta\}^T = \{u, v, w, u_1, v_1\},
\tag{11}
$$

where $[L]$ is a 5×5 matrix of partial differential operators, the components of which are given in Appendix 1.

In the next section, Equations (11) are solved for the flexural free vibrations of angle-ply laminated cylindrical shells subjected to any set of variationally consistent edge boundary conditions. All sets of these boundary conditions that can possibly be applied on either of the two shell edges ($x = 0, L$) are given as follows [11, 22]:

$$
\begin{array}{llll}
u & \text{or} & N_x & \text{prescribed},\\
v & \text{or} & N_{xs} + M_{xs}/R & \text{prescribed},\\
w & \text{or} & M_{x,x} + M_{xs,s} & \text{prescribed},\\
w_{,x} & \text{or} & M_x & \text{prescribed},\\
u_1 & \text{or} & M^a_x & \text{prescribed},\\
v_1 & \text{or} & M^a_{xs} & \text{prescribed},
\end{array}
\tag{12}
$$

and their number (six) makes clear that the set of the Navier-type partial differential equations (11) is of the twelfth order.

3. Flexural vibrations of angle-ply laminated cylindrical shells

For free flexural vibrations of angle-ply laminated circular cylindrical shells, the five unknown displacement functions are expressed in the following form:

$$
\begin{aligned}
u(x,s;t) &= \cos(\omega t)[\overline{u}^{I}(x)\sin(ns/R) + \overline{u}^{II}(x)\cos(ns/R)],\\
v(x,s;t) &= \cos(\omega t)[\overline{v}^{I}(x)\sin(ns/R) + \overline{v}^{II}(x)\cos(ns/R)],\\
w(x,s;t) &= \cos(\omega t)[\overline{w}^{I}(x)\sin(ns/R) + \overline{w}^{II}(x)\cos(ns/R)],\\
u_1(x,s;t) &= \cos(\omega t)[\overline{u}_1^{I}(x)\sin(ns/R) + \overline{u}_1^{II}(x)\cos(ns/R)],\\
v_1(x,s;t) &= \cos(\omega t)[\overline{v}_1^{I}(x)\sin(ns/R) + \overline{u}_1^{II}(x)\cos(ns/R)],
\end{aligned}
\tag{13}
$$

where ω is an unknown natural frequency of vibration and the integer number n is the circumferential full-wave number. Each of these representations can be regarded as a single harmonic in the Fourier-series expansion of the corresponding displacement component around the shell circumference. The x-dependent parts can be regarded as the unknown coefficients in those Fourier-series expansions; they are the ten main unknown functions of the problem considered. On the other hand, the s-dependent parts enable the satisfaction of all the periodicity requirements that should be satisfied around the circumference of the closed cylindrical shell considered.

Upon inserting expressions (13) into Equations (11) and, then, collecting together the coefficients of the sine and the cosine terms, one obtains the two coupled sets (B1) and (B2) of ordinary differential equations with respect to the axial co-ordinate parameter, x (see Appendix 2).

The coupled sets of the differential equations (B1) and (B2) are next solved on the basis of the state space concept. To this end, the following transformations are employed:

$$
\begin{aligned}
&\overline{u}^{I} = Z_1, \quad \overline{u}^{I}_{,x} = Z_2, \quad \overline{v}^{I} = Z_3, \quad \overline{v}^{I}_{,x} = Z_4,\\
&\overline{w}^{I} = Z_5, \quad \overline{w}^{I}_{,x} = Z_6, \quad \overline{w}^{I}_{,xx} = Z_7, \quad \overline{w}^{I}_{,xxx} = Z_8,\\
&\overline{u}^{I}_1 = Z_9, \quad \overline{u}^{I}_{1,x} = Z_{10}, \quad \overline{v}^{I}_1 = Z_{11}, \quad \overline{v}^{I}_{1,x} = Z_{12},\\
&\overline{u}^{II} = Z_{13} \quad \overline{u}^{II}_{,x} = Z_{14}, \quad \overline{v}^{II} = Z_{15}, \quad \overline{v}^{II}_{,x} = Z_{16},\\
&\overline{w}^{II} = Z_{17}, \quad \overline{w}^{II}_{,x} = Z_{18}, \quad \overline{w}^{II}_{,xx} = Z_{19}, \quad \overline{w}^{II}_{,xxx} = Z_{20},\\
&\overline{u}^{II}_1 = Z_{21} \quad \overline{u}^{II}_{1,x} = Z_{22}, \quad \overline{v}^{II}_1 = Z_{23}, \quad \overline{v}^{II}_{1,x} = Z_{24}.
\end{aligned}
\tag{14}
$$

After a considerable amount of algebra [12], these transformations bring Equations (B1) and (B2) into the following matrix form:

$$\{\boldsymbol{Z}'\} = [\boldsymbol{K}]\{\boldsymbol{Z}\}. \tag{15}$$

Moreover, the appearing 24×24 matrix $[\boldsymbol{K}]$ is obtained in the following form:

$$[\boldsymbol{K}] = \begin{bmatrix} \boldsymbol{T} & \boldsymbol{Y} \\ -\boldsymbol{Y} & \boldsymbol{T} \end{bmatrix}, \tag{16}$$

where the nonzero elements of the 12 × 12 submatrices $[\boldsymbol{T}]$ and $[\boldsymbol{Y}]$ are given explicitly in [12] and are dependent on the unknown natural frequency ω. It should be mentioned that in the particular case of cylinders made of one or more specially orthotropic layers, all elements of $[Y]$ become zero, while $[T]$ coincides with the corresponding $12 \times 12[K]$-matrix obtained in [22]. This makes it further clear that the study presented in [22] is a particular case of the present analysis.

In the case that the matrix $[K]$ has twenty four distinct eigenvalues $\lambda_i (i = 1, 2, \ldots, 24)$, the solution of Equation (15) can be written according to,

$$\{\boldsymbol{Z}\} = \mathrm{e}^{[\boldsymbol{K}]x}\{\boldsymbol{Z}_0\}, \quad \mathrm{e}^{[\boldsymbol{K}]x} = [\boldsymbol{Q}][\mathrm{diag}(\mathrm{e}^{\lambda_i x})][\boldsymbol{Q}]^{-1}, \tag{17}$$

where $\{\boldsymbol{Z}_0\}$ is a column vector containing the twenty four arbitrary constants of integration and,

$$[\boldsymbol{Q}] = [\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_{24}], \tag{18}$$

is a 24 × 24 matrix containing the corresponding eigenvectors of $[\boldsymbol{K}]$. Hence, by denoting $\{\boldsymbol{c}\} = [\boldsymbol{Q}]^{-1}\{\boldsymbol{Z}_0\}$, we can alternatively express the solution (17) as,

$$\{\boldsymbol{Z}\} = \sum_{i=1}^{24} c_i \mathrm{e}^{\lambda_i x}. \tag{19}$$

The appearing arbitrary constant will be determined when a set of boundary conditions will be imposed on the shell edges.

In this paper the following sets of simply supported (S) and clamped (C) boundary conditions will be imposed on the two shell edges $(x = 0, L)$:

$$\begin{aligned} S: &\quad N_x = v = w = M_x = M_x^a = v_1 = 0, \\ C: &\quad u = v = w = w_{,x} = u_1 = v_1 = 0. \end{aligned} \tag{20}$$

If either of these homogeneous sets of boundary conditions are used, the solutions (19) yields twenty four simultaneous homogeneous linear algebraic equations, which can be written in the following form:

$$[\boldsymbol{D}]\{\boldsymbol{c}\} = \{\boldsymbol{0}\}. \tag{21}$$

Hence, the natural frequencies of vibration are determined to be those values of ω that nullify the determinant of the matrix $[\boldsymbol{D}]$. This can be achieved when this determinant is treated as a function, of ω, the roots of which can be sought by standard numerical analysis methods. To this end, all results presented in Section 5, below, were obtained by standard numerical routines of the NAG library associated, appropriately, to a relevant FORTRAN computer programme.

4. Axisymmetric vibrations of angle-ply laminated cylindrical shells

The axisymmetric free vibrations of circular cylindrical shells are defined as those such vibrations whose pattern is independent of the circumferential co-ordinate parameter, s [4, 13, 14]. Hence, when all the terms that involve differentials with respect to s are dropped, the formulation of this axisymmetric vibration problem is obtained as a particular case of the formulation presented in Section 2. It is of interest to note that the fifth, the eighth and the ninth elements of the column vector that appears in the right-hand side of the constitutive Equation (6a) are consequently dropped. Hence, the fifth, the eighth and the ninth columns should be removed from the appearing rigidities matrix, which becomes therefore a 10×7 matrix.

In the particular case of an isotropic, an orthotropic or a cross-ply laminated cylinder, this simplification of axially symmetric motions results in the uncoupling of the equations that govern the longitudinal and the torsional vibrations (see, for instance, reference [4]). Hence, the two problems occur as being physically uncoupled and are studied separately, by the solution of two separate sets of differential equations. In more detail, the longitudinal vibrations are described by a set of three Navier-type differential equations, with main unknowns the displacement components u, w and u_1, while the torsional vibrations are described by a corresponding set of two equations, with main unknowns the displacement components v and v_1. In the present case, however, the monoclinic constitution of the layers has caused the coupling of the longitudinal and the torsional motions. Hence, despite the axisymmetric vibrations simplification, all five Navier-type equations of motion remain coupled and should be solved simultaneously. Their solution can, however, be obtained as a particular case of the flexural vibration solution presented in the preceding section. Moreover, the fact that the above mentioned classes of cylindrical-shell vibrations are obtained as particular cases of the present analysis assisted the authors substantially in checking and verifying the correctness of the algebra involved. In the same context, the present axisymmetric vibration solution assisted in checking the correctness of the algebra involved in the case of the most general flexural vibrations discussed in Section 3.

Defining the axisymmetric vibrations of cylindrical shells as being independent of the circumferential co-ordinate parameter is essentially equivalent with setting the circumferential wave number of the flexural vibrations, n, equal to zero. As a result, the sine terms are dropped in the displacement model (13), the spatial part of which involves finally only five unknown functions of the axial co-ordinate parameter, x. These are essentially the ones denoted with a superscript ^{II}. With a 90° rotation of the co-ordinate origin, around the shell circumference (the equivalent transformation is $s \to s - R\pi/2$), the functions denoted with a superscript ^{I} could be employed as the five main unknowns of the problem. Hence, the superscript becomes redundant and should be dropped from the unknown displacement function.

Upon inserting such a displacement model into the axisymmetric version of the Equations (11), one obtains a set of five ordinary differential equations with respect to the axial co-ordinate parameter, x. This can alternatively be obtained as a particular case of either Equations (A4a) or Equations (A4b) (see Appendix) upon (i) dropping all the P_i coefficients, as well as those K_i coefficients that have n as a common factor, and (ii) simplifying the remaining K_i coefficients, by setting $n = 0$ where appropriate. For an application of the state space concept, the thus obtained set of ordinary differential equations is brought into the form (15), in which the elements of the column matriz $\{\boldsymbol{Z}\}$ represent now the first twelve (or the second twelve) of the transformations (14). Moreover, $[\boldsymbol{K}]$ is now a 12×12 matrix. Its elements are given explicitly in [12] and can alternatively be obtained upon setting $n = 0$ in the corresponding elements of the 12×12 submatrix $[\boldsymbol{T}]$ that appears in Equation (16) ($[\boldsymbol{Y}] = \boldsymbol{0}$). The remaining of the axisymmetric vibrations solution follows then the lines of the

Table 1. Lowest axisymmetric frequency parameter, ω^*, of eight-layered SS cylinders with a symmetric angle-ply lay-up of the form $[(\pm\theta)_2]_s (h/R = 0{\cdot}3, L/R = 1)$.

	Present work			Ref [14]
θ	$2D$	$2D$	$2D$	$3D$
(degrees)	USDT	PSD	HSDT	Elasticity
0°	1·0000	0·9604	0·9600	0·9531
15°	0·9865	0·9460	0·9456	0·9292
30°	0·9540	0·9140	0·9137	0·8739
45°	0·9185	0·8662	0·8660	0·8327
60°	1·0270	1·0230	1·0230	0·9938
75°	1·2670	1·2670	1·2670	1·2672
90°	1·0000	1·0000	1·0000	1·0000

Equations (17–21) in which, however, the maximum number of the distinct eigenvalues of the matrix $[\boldsymbol{K}]$ should be dropped, from twenty four to twelve.

5. Numerical results and discussion

As has already been mentioned, all numerical results presented in this section are for cylinders having both their edges simply supported (SS cylinders) or clamped (CC cylinders). Most of the results shown were based on choices of the shape functions, $\Phi_1(z)$ and $\Phi_2(z)$, that are consistent with the so-called parabolic shear-deformable shell theory (PSDT). For comparison purposes, however, two more choices of the shape functions are also used. These are consistent with the so-called uniform shear-deformable theory (USDT) and a shear-deformable theory (HSDT) that uses shape functions of hyperbolic type [23]. In more detail, the shape functions employed for each theory are as follows:

$$\begin{aligned} &\text{USDT:}\quad \Phi_1(z) = \Phi_2(z) = z,\\ &\text{PSDT:}\quad \Phi_1(z) = \Phi_2(z) = z(1 - 4z^2/h^2),\\ &\text{HSDT:}\quad \Phi_1(z) = \Phi_2(z) = h\ \sinh(z/h) - z\ \sinh(1/2). \end{aligned} \tag{22}$$

In some cases, which involve zero shape functions, the classical Love-type shell theory (CST) has also been used for comparison purposes.

5.1. Axisymmetric vibrations ($n = 0$)

For a certain family of eight-layered symmetric angle-ply laminated SS cylinders, Table 1 compares the lowest axisymmetric frequency parameter,

$$\omega^* = (\omega R/\pi)\sqrt{\rho/C_{66}}, \tag{23}$$

obtained on the basis of the present shell-theory analysis, with the corresponding frequency parameter obtained in [14] on the basis of exact three-dimensional elasticity analysis. The shells are particularly thick ($h/R = 0{\cdot}3$) and highly reinforced, with every layer having the following material properties,

$$E_L/E_T = 40, \quad G_{LT}/E_T = 0{\cdot}6, \quad G_{TT}/E_T = 0{\cdot}5, \quad \nu_{LT} = \nu_{TT} = 0{\cdot}25. \tag{24}$$

The frequency parameters are presented for different fibre orientations, θ.

For so thick and highly reinforced cylinders, one should not expect from a two-dimensional shell theory to produce accurate predictions of the natural frequencies and vibrations. However, it is observed that, despite the high reinforcement and thickness, the frequency predictions that are based on either PSDT or HSDT are always reasonably close to the corresponding elasticity predictions. Indeed, even the highest discrepancy, which is about 4·5% and occurs when the fibres are oriented between $\theta = 30°$ and $\theta = 45°$, is still within the engineering acceptable limit (5%). For smaller and, particularly, for higher values of θ, these discrepancies decrease continuously and take their smallest values for $\theta = 0°$ and $\theta = 90°$, respectively, that is when the material of the cylinder becomes homogeneous orthotropic. It is of particular interest to notice that these discrepancies become essentially negligible for very large values of θ, that is when the reinforcement becomes predominantly circumferential. They indeed disappear completely for $\theta = 90°$, that is when the high circumferential reinforcement is expected to restrict considerably the axially symmetric motion of the cylinder. It should be noted, in this respect, that even the USDT predicts the exact vibration frequency in this later case. When the value of θ is decreased, the USDT predictions become increasingly inaccurate, with an exception the $\theta = 0°$ case, in which the obtained frequency is, however, still about 5% higher than its exact value.

Under these considerations, the USDT theory will not be used any further in this study. Moreover, the HSDT seems to give always slightly better frequency predictions than the PSDT does, but the improvement of the corresponding numerical results is practically negligible. It is therefore concluded that there is no practical need of replacing the well-established shape functions of the PSDT with the rather complicated shape functions of the HSDT [see Equations (22)]. Hence, only numerical results based on the PSDT will be shown in what follows.

It is of interest to notice, that the shape functions of all three versions of the shear-deformable theory employed assume, erroneously, that the interlaminar stresses are discontinuous at the material interfaces of the laminate. This should be directly connected with the fact that, for the axisymmetric vibration problem considered, the relative errors of all three versions are magnified between $\theta = 30°$ and $\theta = 45°$. Namely, in cases that the discontinuity of the interlaminer stresses is magnified considerably while, at the same time, the circumferential reinforcement is not high enough to restrict considerably the axisymmetric motion of the cylinder. It is therefore concluded that a further improvement of the theory is needed with regard to the implementation of new features and shape functions that could guarantee the through-thickness continuity of interlaminar stresses in angle-ply laminates. It should be noted, however, that this task is not as straightforward as it appears to be in the case of cross-ply laminates [11, 12, 22] and, as such, it has been left as a subject for future investigation.

Table 2 shows the first three axisymmetric frequency parameters,

$$\overline{\omega} = 100\omega h\sqrt{\rho/E_T}, \tag{25}$$

Table 2. First three axisymmetric natural frequency parameters, $\overline{\omega}$, of four-layered cylinders having a symmetric lay-up of the form $[+\theta/-\theta]_s\,(h/R = 0{\cdot}01,\ L/R = 0{\cdot}1)$.

Boundary conditions	θ (degrees)	Vibration mode I	II	III
SS	0°	11·700	22·211	33·261
	45°	6·825	20·856	36·952
	90°	5·700	11·433	21·565
CC	0°	17·880	22·214	37·540
	45°	11·274	24·205	39·815
	90°	7·640	15·130	22·214

for three values of the fibre-orientation angle of a four-layered symmetrically laminated cylindrical shell subjected to both SS and CC edge boundary conditions. The shell is a thin one $(h/R = 0{\cdot}01)$. The material properties in each of its layers are as follows:

$$E_L/E_T = 25, \quad G_{LT}/E_T = 0{\cdot}5, \quad G_{TT}/E_T = 0{\cdot}2, \quad \upsilon_{LT} = \upsilon_{TT} = 0{\cdot}25. \tag{26}$$

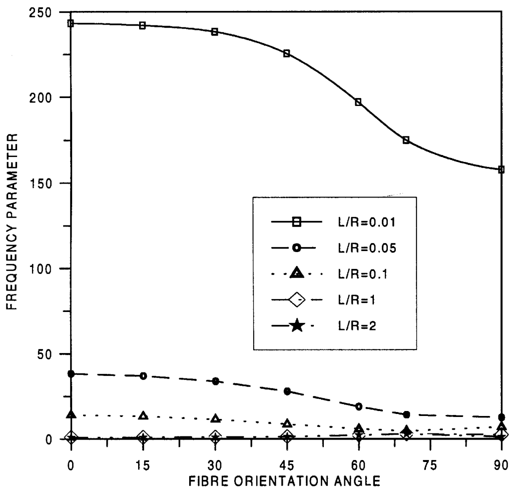

Figure 1. Variation of the lowest axisymmetric frequency parameter $\overline{\omega}$, as a function of θ, for several L/R values of a SS cylindrical shell having a $[+\theta/-\theta]_s$ lay-up $(h/R = 0{\cdot}1)$.

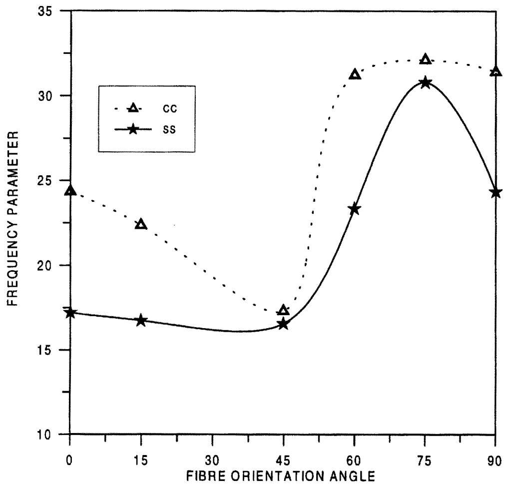

Figure 2. Variation of the lowest axisymmetric frequency parameter $\overline{\omega}$, as a function of θ, for SS and CC cylindrical shell having a $[+\theta/-\theta]_s$ lay-up $(L/R = 1,\ h/R = 0{\cdot}1)$.

For $\theta = 0°$ or $\theta = 90°$ the fibres are all aligned to the same direction forming, thus, a homogeneous orthotropic shell. As was expected, the frequencies of CC shells are always higher than the corresponding frequencies of SS shells. It is of interest to note that, for this type of thin and short shells $(L/R = 0{\cdot}1)$, the $[45°/-45°]_s$ arrangement yields lower frequency parameters than the 90° homogeneous case, regardless of the edge boundary conditions employed.

Table 3. First five natural frequency parameters, $\hat{\omega}$, of four-layered cylinders having a symmetric $[+30°/-30°]_s$ lay-up ($h/R0{\cdot}01$, $L/R = 4$).

Mode	SS			CC		
	Ref [17]	CST	PSDT	Ref [17]	CST	PSDT
I	0·1232	0·1233^4	0·1232	0·1827	0·1818^5	0·1817
II	0·1314	0·1315^5	0·1313	0·1925	0·1920^6	0·1918
III	0·1675	0·1674^6	0·1753	0·2212	0·2198^4	0·2195
IV	0·1753	0·1755^3	0·1672	0·2295	0·2293^7	0·2285
V	0·2160	0·2160^7	0·2150	0·2811	0·2802^8	0·2799

For SS shells having the same thickness to radius ratio ($h/R = 0{\cdot}01$), as well as the same $[+\theta/-\theta]_s$ material arrangement and properties, Figure 1 shows the variation of the lowest axisymmetric frequency parameter, $\overline{\omega}$, as a function of both the fibre orientation and the length to the radius ratio. It is observed that shells with $L/R = 1$ and $L/R = 2$ vibrate with very similar lowest frequencies for all fibre orientation angles. Hence, as far as axisymmetric vibrations of such thin laminates are concerned, the bound between angle-ply shells of finite and practically infinite extent should not be placed much further than $L/R = 1$. When the L/R ratio is decreased, the frequency parameters increase. It can further be concluded that, for this type of thin shells, the lowest axisymmetric frequency decreases when the fibre orientation angle is increased from 0° to 90°. As has already been seen, this situation differs considerably from that of corresponding thick angle-ply laminated shells, for which the lowest axisymmetric frequency occurs at about $\theta = 45°$.

This later observation is further justified from the results shown in Figure 2. There, for moderately thick shells ($h/R = 0{\cdot}1$, $L/R = 1$) having either SS or CC edge boundaries, the variation of the lowest axisymmetric vibration frequency is shown as a function of the fibre-orientation angle. Despite the fact that these shells are not as thick as those considered in Table 1, the maximum axisymmetric vibration frequency occurs just above $\theta = 45°$, regardless of the type of the edge boundary condition. As was expected, CC shells always vibrate with higher frequencies than corresponding SS shells, but corresponding frequency values appear to approach considerably towards the afore-mentioned minima of the two curves ($\theta = 45°$) as well as at their corresponding maxima ($\theta = 75°$).

5.2. FLEXURAL VIBRATIONS ($n \neq 0$)

As has already been mentioned in the Introduction, the study of Narita *et al.* [17] on flexural vibrations of angle-ply laminated shells contains a substantial amount of relevant numerical results. From this point of view, it is perhaps the most complete study that has appeared so far on this subject. The model adopted in [17] was, however, based on a second-approximation classical shell theory of the Flugge-type [18] and the analysis was based on the application of the Ritz method to the corresponding energy functional. Due to the extensive amount of results presented and discussed in [17], it was found more interesting to compare corresponding results based on the two different approaches (the state space concept and the Ritz method [17]) than to produce and discuss new results in this study. The good agreement that was always

Table 4. First five natural frequency parameters, $\hat{\omega}$, of four-layered cylinders having a symmetric $[+45°/-45°]_s$ lay-up ($h/R = 0{\cdot}01$, $L/R = 4$).

	SS			CC		
Mode	Ref [17]	CST	PSDT	Ref [17]	CST	PSDT
I	0·1193	$0{\cdot}1195^3$	0·1190	0·1789	$0{\cdot}1760^4$	0·1757
II	0·1253	$0{\cdot}1254^4$	0·1250	0·1951	$0{\cdot}1937^5$	0·1932
III	0·1733	$0{\cdot}1731^5$	0·1728	0·2444	$0{\cdot}2383^3$	0·2380
IV	0·2204	$0{\cdot}2204^2$	0·2200	0·2474	$0{\cdot}2464^6$	0·2450
V	0·2362	$0{\cdot}2355^6$	0·2350	0·3168	$0{\cdot}3151^7$	0·3145

Table 5. First five natural frequency parameters, $\hat{\omega}$, of four-layered cylinders having a symmetric $[+60°/-60°]_s$ lay-up ($h/R = 0{\cdot}01$, $L/R = 4$).

	SS			CC		
Mode	Ref [17]	CST	PSDT	Ref [17]	CST	PSDT
I	0·1093	$0{\cdot}1094^3$	0·1093	0·1796	$0{\cdot}1780^4$	0·1775
II	0·1533	$0{\cdot}1532^3$	0·1530	0·1796	$0{\cdot}1780^4$	0·1775
III	0·1533	$0{\cdot}1532^4$	0·1530	0·1894	$0{\cdot}1850^3$	0·1850
IV	0·2283	$0{\cdot}1280^5$	0·2271	0·3136	$0{\cdot}3110^5$	0·3100
V	0·2392	$0{\cdot}2400^4$	0·2390	0·3196	$0{\cdot}3144^4$	0·3144

observed was very convincing for the reliability of both approaches. This is demonstrated in Tables 3–5, where some comparisons are shown of corresponding numerical results.

Tables 3–5 compare the first five flexural frequency parameters,

$$\hat{\omega} = \omega R\sqrt{\rho/E_T}, \tag{27}$$

of four-layered thin cylinders ($h/R = 0{\cdot}01$, $L/R = 4$) having both SS and CC edge boundaries and a symmetric lay up of the form $[+\theta/-\theta]_s$. Each layer has the following material properties,

$$E_L/E_T = 20, \quad G_{LT}/E_T = 0{\cdot}65, \quad G_{TT}/E_T = 0{\cdot}2, \quad \nu_{LT} = \nu_{TT} = 0{\cdot}25, \tag{28}$$

while the fibre orientation angle varies from 30° to 60°. Both CST and PSDT have been used for the results obtained on the basis of the state space concept. In all cases considered, both the frequencies and the circumferential mode numbers (given as superscripts of the CST results) were found to be in excellent agreement with the corresponding results due to Narita *et al.* [17]. As was expected, however, the PSDT frequency parameters were always slightly lower than those predicted by either of the classical shell theories, regardless of the type of the edge boundary conditions. It is expected that, when either the shell thickness or the material reinforcement is increased, this difference observed between corresponding frequencies based on CST and PSDT will increase, with the results of PSDT being always closer to the exact results that could be obtained by means of a perspective solution of the corresponding equation of three-dimensional elasticity.

6. Conclusions

This paper studies both the axisymmetric and the flexural vibrations of finite, closed, circular cylindrical shells, made of one or more monoclinic layers and subjected to different sets of edge boundary conditions. The study and the analysis were based on the Love-type version of a unified shear-deformable shell theory and the state space concept, respectively. The state space concept is an exact method for solving ordinary differential equations with constant coefficients. In dealing with free vibrations of finite, closed, cylindrical shells, the method has not been applied previously for a more complicated material arrangement than that of a cross-ply lay-up.

As was reported in previous studies that dealt with vibrations of cross-ply laminates, the straightforward application of the method may cause severe numerical instabilities. It may be noted, in this respect, that neither in the present investigation, in which the angle-ply lay-up has essentially doubled the size of the frequency eigendeterminant, the numerical behaviour of the solution was found to be always stable. If necessary, however, some modified versions of the method that eliminated the numerical instabilities for cross-ply laminates [10, 24, 25] could be extended and used in order to further take care of any similar instabilities involved in the free vibration analysis of angle-ply laminated cylinders.

The unified theoretical formulation employed allowed the comparison of numerical results that were based on several types of shear-deformable shell theories, with corresponding results based on a three-dimensional solution. Among these shear-deformable theories employed, the uniform shear-deformable theory provided the poorest frequency predictions. The well-known parabolic shear-deformable shell theory did not yield the best frequency predictions, but its results were always very close to the best shell-theory predictions obtained. As a result, it was concluded that there was not an immediate practical need for dismissing it or replacing it.

It was pointed out, however, that all the shear-deformable theories employed assumed, erroneously, that the interlaminar stresses are discontinuous at the material interfaces of the angle-ply laminate. This was further connected with the fact that the relative errors, with respect to the corresponding three-dimensional elasticity results, were magnified considerably for certain combinations of the material reinforcement and the fibre angle. It was therefore concluded that a further investigation is needed towards the improvement of the theory. This should be dealing with the implementation of new features and shape functions that could guarantee the through-thickness continuity of interlaminar stresses in angle-ply laminates. The implementation of these features, which is already available in cross-ply laminates, is not straightforward for angle-ply laminates. It may cause an increase in the number of the degrees of freedom involved in the theoretical modelling, with corresponding consequences being transferred onto the analytical and the numerical treatment of the problem considered. Hence, it was suggested and left as a subject for future research investigations.

Appendix A

The components of the 5×5 matrix $[\boldsymbol{L}]$ appearing in Equation (11) are given as follows:

$$
\begin{aligned}
L_{11} &= A_{11}()_{,xx} + 2A_{16}()_{,xs} + A_{66}()_{,ss} - \rho_0()_{,tt,}\\
L_{12} &= a_{16}()_{,xx} + (a_{12} + a_{66})()_{,xs} + a_{26}()_{,ss},\\
L_{13} &= -B_{11}()_{,xxx} - 3B_{16}()_{,xxs} + b_1()_{,xss} - B_{26}()_{,sss}\\
&\quad +A_{12}()_{,x}/R + A_{26}()_{,s}/R + \rho_1()_{,xtt},\\
L_{14} &= B'_{111}()_{,xx} + 2B'_{161}()_{,xs} + B'_{661}()_{,ss} - \rho_0^{-11}()_{,tt},\\
L_{15} &= B'_{162}()_{,xx} + b'_2()_{,xs} + B'_{262}()_{,ss},\\
L_{22} &= \bar{a}_{66}()_{,xx} + 2\bar{a}_{26}()_{,xs} + \bar{a}_{22}()_{,ss} - (\rho_0 + 2\rho_1/R + \rho_2/R^2)()_{,tt},\\
L_{23} &= -b_{16}()_{,xxx} - (2b_{66} + b_{12})()_{,xxs} - 3b_{26}()_{,xss} - b_{22}()_{,sss}\\
&\quad +a_{26}()_{,x}/R + a_{22}()_{,s}/R + (\rho_2/R - \rho_1)()_{,stt},\\
L_{24} &= b'_{161}()_{,xx} + (b'_{121} + b'_{661})()_{,xs} + b'_{261}()_{,ss},\\
L_{25} &= b'_{662}()_{,xx} + 2b'_{262}()_{,xs} + b'_{222}()_{,ss} - (\bar{\rho}_0^{21} + \bar{\rho}_1^{21}/R)()_{,tt},\\
L_{33} &= D_{11}()_{,xxxx} + D_{22}()_{,ssss} + 4D_{16}()_{,xxxs} + 2d_1()_{,xxss} + 4D_{26}()_{,xsss}\\
&\quad -2B_{12}()_{,xx}/R - 4B_{26}()_{,xs}/R - 2B_{22}()_{,ss}/R + A_{22}()/R^2\\
&\quad +\rho_0()_{,tt} - \rho_2()_{,sstt} - \rho_2()_{,xxtt},\\
L_{34} &= -D_{111}()_{,xxx} - 3D_{161}()_{,xxs} - d'_1()_{,xss} - D_{261}()_{,sss}\\
&\quad +B_{121}()_{,x}/R + B_{261}()_{,s}/R + \bar{\rho}_1^{11}()_{,xtt},\\
L_{35} &= -D_{162}()_{,xxx} - 3D_{262}()_{,xss} - d'_2()_{,xxs} - D_{222}()_{,sss}\\
&\quad +B_{262}()_{,x}/R + B_{222}()_{,s}/R + \bar{\rho}_1^{21}()_{,stt},\\
L_{44} &= D_{1111}()_{,xx} + D_{6611}()_{,ss} + D_{1611}()_{,xs} - A_{5511}() - \bar{\rho}_0^{21}()_{,tt},\\
L_{45} &= D_{1612}()_{,xx} + D_{2612}()_{,ss} + d''_3()_{,xs} - A_{4512}(),\\
L_{55} &= D_{6622}()_{,xx} + D_{2222}()_{,ss} + 2D_{2622}()_{,xs} - A_{4422}() - \bar{\rho}_0^{22}()_{,tt},
\end{aligned}
\tag{A1}
$$

where

$$
\begin{aligned}
&(a_{ij}, b_{ij}) = [(A_{ij} + B_{ij}/R), (B_{ij} + D_{ij}/R)],\\
&b'_{ijk} = B_{ijk} + D_{ijk}/R, \bar{a}_{ij} = a_{ij} + b_{ij}/R,
\end{aligned}
\tag{A2}
$$

and

$$
\begin{aligned}
b_1 &= B_{12} + 2B_{66}, d_1 = D_{12} + 2D_{66},\\
&\quad b'_i = B_{12i} + 2B_{66i}, d'_i = D_{12i} + 2D_{66i}, d''_3 = D_{1212} + D_{6612}.
\end{aligned}
\tag{A3}
$$

Appendix B

Upon inserting expressions (13) into Equations (11) and then collecting the coefficients of the sine and the cosine terms, one obtains two coupled sets of ordinary differential equations with respect to the axial coordinate parameter x. The first set, obtained by setting the coefficients of the sine terms equal to zero, can be brought into the following form:

$$
\begin{aligned}
\bar{u}^I_{,xx} &= K_1\bar{u}^I + K_2\bar{v}^I + K_3\bar{v}^I_{,xx} + K_4\bar{w}^I_{,x} + K_5\bar{w}^I_{,xxx} + K_6\bar{u}^I_1 + K_7\bar{u}^I_{1,xx}\\
&\quad + K_8\bar{v}^I_1 + K_9\bar{v}^I_{1,xx} + P_1\bar{u}^{II}_{,x} + P_2\bar{v}^{II}_{,x} + P_3\bar{w}^{II} + P_4\bar{w}^{II}_{,xx} + P_5\bar{u}^{II}_{1,x} + P_6\bar{v}^{II}_{1,x},\\
\bar{v}^I_{,xx} &= K_{10}\bar{u}^I + K_{11}\bar{u}^I_{,xx} + K_{12}\bar{v}^I + K_{13}\bar{w}^I_{,x} + K_{14}\bar{w}^I_{,xxx} + K_{15}\bar{u}^I_1 + K_{16}\bar{u}^I_{1,xx}\\
&\quad + K_{17}\bar{v}^I_1 + K_{18}\bar{v}^I_{1,xx} + P_7\bar{u}^{II}_{,x} + P_8\bar{v}^{II}_{,x} + P_9\bar{w}^{11} + P_{10}\bar{w}^{II}_{,xx} + P_{11}\bar{u}^{II}_{1,x} + P_{12}\bar{v}^{II}_{1,x},\\
\bar{w}^I_{,xxxx} &= K_{19}\bar{u}^I_{,x} + K_{20}\bar{u}^I_{,xxx} + K_{21}\bar{v}^I_{,x} + K_{22}\bar{v}^I_{,xxx} + K_{23}\bar{w}^I + K_{24}\bar{w}^I_{,xxx} + K_{25}\bar{u}^I_{1,x}\\
&\quad + K_{26}\bar{u}^I_{1,xxx} + K_{27}\bar{v}^I_{1,x} + K_{28}\bar{v}^I_{1,xxx} + P_{13}\bar{u}^{II} + P_{14}\bar{u}^{II}_{,xx} + P_{15}\bar{v}^{II} + P_{16}\bar{v}^{II}_{,xx} \qquad \text{(B1)}\\
&\quad + P_{17}\bar{w}^{II}_{,x} + P_{18}\bar{w}^{II}_{,xxx} + P_{19}\bar{u}^{II}_1 + P_{20}\bar{u}^{II}_{1,xx} + P_{21}\bar{v}^{II}_1 + P_{22}\bar{v}^{II}_{1,xx},\\
\bar{u}^I_{1,xx} &= K_{29}\bar{u}^I + K_{30}\bar{u}^I_{,xx} + K_{31}\bar{v}^I + K_{32}\bar{v}^I_{,xx} + K_{33}\bar{w}^I_{,x} + K_{34}\bar{w}^I_{,xxx} + K_{35}\bar{u}^I_1 + K_{36}\bar{u}^I_{1,xx}\\
&\quad + K_{37}\bar{v}^I_{1,xx} + P_{23}\bar{u}^{II}_{,x} + P_{24}\bar{v}^{II}_{,x} + P_{25}\bar{w}^{II} + P_{26}\bar{w}^{II}_{,xx} + P_{27}\bar{u}^{II}_{1,x} + P_{28}\bar{v}^{II}_{1,x},\\
\bar{v}^I_{1,xx} &= K_{38}\bar{u}^I + K_{39}\bar{u}^I_{,xx} + K_{40}\bar{v}^I + K_{41}\bar{v}^I_{,xx} + K_{42}\bar{w}^I_{,x} + K_{43}\bar{w}^I_{,xxx} + K_{44}\bar{u}^I_1 + K_{45}\bar{u}^I_{1,xx}\\
&\quad + K_{46}\bar{v}^I_1 + P_{29}\bar{u}^{II}_{,x} + P_{30}\bar{v}^{II}_{,x} + P_{31}\bar{w}^{II} + P_{32}\bar{w}^{II}_{,xx} + P_{33}\bar{u}^{II}_{1,x} + P_{34}\bar{v}^{II}_{1,x},
\end{aligned}
$$

while the second set is given as follows:

$$
\begin{aligned}
\bar{u}^{II}_{,xx} &= K_1\bar{u}^{II} + K_2\bar{v}^{II} + K_3\bar{v}^{II}_{,xx} + K_4\bar{w}^{II}_{,x} + K_5\bar{w}^{II}_{,xxx} + K_6\bar{u}^{II}_1 + K_7\bar{u}^{II}_{1,xx}\\
&\quad + K_8\bar{v}^{II}_1 + K_9\bar{v}^{II}_{1,xx} - (P_1\bar{u}^I_{,x} + P_2\bar{v}^I_{,x} + P_3\bar{w}^I + P_4\bar{w}^I_{,xx} + P_5\bar{u}^I_{1,x} + P_6\bar{v}^I_{1,x}),\\
\bar{v}^{II}_{,xx} &= K_{10}\bar{u}^{II} + K_{11}\bar{u}^{II}_{,xx} + K_{12}\bar{v}^{II} + K_{13}\bar{w}^{II}_{,x} + K_{14}\bar{w}^{II}_{,xxx} + K_{15}\bar{u}^{II}_1 + K_{16}\bar{u}^{II}_{1,xx}\\
&\quad + K_{17}\bar{v}^{II}_1 + K_{18}\bar{v}^{II}_{1,xx} - (P_7\bar{u}^I_{,x} + P_8\bar{v}^I_{,x} + P_9\bar{w}^I + P_{10}\bar{w}^I_{,xx} + P_{11}\bar{u}^I_{1,x} + P_{12}\bar{v}^I_{1,x}),\\
\bar{w}^{II}_{,xxxx} &= K_{19}\bar{u}^{II}_{,x} + K_{20}\bar{u}^{II}_{,xxx} + K_{21}\bar{v}^{II}_{,x} + K_{22}\bar{v}^{II}_{,xxx} + K_{23}\bar{w}^{II} + K_{24}\bar{w}^{II}_{,xxx} + K_{25}\bar{u}^{II}_{1,x}\\
&\quad + K_{26}\bar{u}^{II}_{1,xxx} + K_{27}\bar{v}^{II}_{1,x} + K_{28}\bar{v}^{II}_{1,xxx} - (P_{13}\bar{u}^I + P_{14}\bar{u}^I_{,xx} + P_{15}\bar{v}^I + P_{16}\bar{v}^I_{,xx} \qquad \text{(B2)}\\
&\quad + P_{17}\bar{w}^I_{,x} + P_{18}\bar{w}^I_{,xxx} + P_{19}\bar{u}^I_1 + P_{20}\bar{u}^I_{1,xx} + P_{21}\bar{v}^I_1 + P_{22}\bar{v}^I_{1,xx}),\\
\bar{u}^{II}_{1,xx} &= K_{29}\bar{u}^{II} + K_{30}\bar{u}^{II}_{,xx} + K_{31}\bar{v}^{II} + K_{32}\bar{v}^{II}_{,xx} + K_{33}\bar{w}^{II}_{,x} + K_{34}\bar{w}^{II}_{,xxx} + K_{35}\bar{u}^{II}_1 + K_{36}\bar{u}^{II}_{1,xx}\\
&\quad + K_{37}\bar{v}^{II}_{1,xx} - (P_{23}\bar{u}^I_{,x} + P_{24}\bar{v}^I_{,x} + P_{25}\bar{w}^I + P_{26}\bar{w}^I_{,xx} + P_{27}\bar{u}^I_{1,x} + P_{28}\bar{v}^I_{1,x}),\\
\bar{v}^{II}_{1,xx} &= K_{38}\bar{u}^{II} + K_{39}\bar{u}^{II}_{,xx} + K_{40}\bar{v}^{II} + K_{41}\bar{v}^{II}_{,xx} + K_{42}\bar{w}^{II}_{,x} + K_{43}\bar{w}^{II}_{,xxx} + K_{44}\bar{u}^{II}_1 + K_{45}\bar{u}^{II}_{1,xx}\\
&\quad + K_{46}\bar{v}^{II}_1 - (P_{29}\bar{u}^I_{,x} + P_{30}\bar{v}^I_{,x} + P_{31}\bar{w}^I + P_{32}\bar{w}^I_{,xx} + P_{33}\bar{u}^I_{1,x} + P_{34}\bar{v}^I_{1,x}).
\end{aligned}
$$

Evidently, the appearing coefficients K_i and P_i, are the same constants in both sets of equations. These are given below. It should be emphasized that Equations (B2) can be obtained from Equations (B1) by interchanging the superscripts ^{I} and ^{II} in the unknown functions involved and by changing the signs of all coefficients P_i.

The constant coefficients that appear in both sets of Equations (B1) and (B2) are given as follows:

1st Equation:

$$
\begin{aligned}
K_1 &= -(A_{66}n_2 + \rho_0\omega^2)/A_{11}, & K_2 &= -a_{26}n_2/A_{11},\\
K_3 &= -a_{16}/A_{11}, & K_4 &= -(A_{12}/R - b_1 n_2 - \rho_1\omega^2)/A_{11},\\
K_5 &= B_{11}/A_{11}, & K_6 &= (B_{661}n_2 + \bar{\rho}_0^{11}\omega^2)/A_{11},\\
K_7 &= -B_{111}/A_{11}, & K_8 &= -B_{262}n_2/A_{11},\\
K_9 &= -B_{162}/A_{11}, & P_1 &= -2A_{16}n_1/A_{11},\\
P_2 &= -(a_{12} + a_{66})n_1/A_{11}, & P_3 &= -(A_{26}n_1/R - B_{26}n_3)/A_{11},\\
P_4 &= 3B_{16}n_1\big/A_{11}, & P_5 &= -2B_{161}n_1/A_{11},\\
P_6 &= -b_2'n_1/A_{11}; & &
\end{aligned}
\tag{B3}
$$

2nd Equation:

$$
\begin{aligned}
K_{10} &= -a_{26}n_2/\bar{a}_{66}, & K_{11} &= -a_{16}/\bar{a}_{66},\\
K_{12} &= -(\bar{a}_{22}n_2 + I_{21}\omega^2)/\bar{a}_{66}, & K_{13} &= -(a_{26}/R - 3b_{26}n_2)/\bar{a}_{66},\\
K_{14} &= b_{16}/\bar{a}_{66}, & K_{15} &= -b_{261}'n_2/\bar{a}_{66},\\
K_{16} &= -b_{161}'/\bar{a}_{66}, & K_{17} &= -(b_{222}'n_2 + I_{23}\omega^2)/\bar{a}_{66},\\
K_{18} &= -b_{662}'/\bar{a}_{66}, & P_7 &= -(a_{12} + a_{66})n_1/\bar{a}_{66},\\
P_8 &= -2\bar{a}_{26}n_1/\bar{a}_{66}, & P_9 &= -[a_{22}n_1/R - b_{22}n_3 + I_{22}n_1\omega^2]/\bar{a}_{66},\\
P_{10} &= (2b_{66} + b_{12})n_1/\bar{a}_{66}, & P_{11} &= -(b_{121}' + b_{661}')n_1/\bar{a}_{66},\\
P_{12} &= -2b_{262}'n_1/\bar{a}_{66}; & &
\end{aligned}
\tag{B4}
$$

3rd Equation:

$$
\begin{aligned}
K_{19} &= (b_1n_2 - A_{12}/R + \rho_1\omega^2)/D_{11}, & K_{20} &= B_{11}/D_{11},\\
K_{21} &= (3b_{26}n_2 - a_{26}/R)/D_{11}, & K_{22} &= b_{16}/D_{11},\\
K_{23} &= (2B_{22}n_2/R - D_{22}n_4 - A_{22}/R^2 + I_{20}\omega^2)/D_{11}, & K_{24} &= (-2d_1n_2 + 2B_{12}/R - \rho_2\omega^2)/D_{11},\\
K_{25} &= (d_1'n_2 - B_{121}/R + \bar{\rho}_1^{II}\omega^2)/D_{11}, & K_{26} &= D_{111}/D_{11},\\
K_{27} &= (3D_{262}n_2 - B_{262}/R)/D_{11}, & K_{28} &= D_{162}/D_{11},\\
P_{13} &= (B_{26}n_3 - A_{26}n_1/R)/D_{11}, & P_{14} &= 3B_{16}n_1/D_{11},\\
P_{15} &= [b_{22}n_3 - a_{22}n_1/R - I_{22}\omega^2n_1]/D_{11}, & P_{16} &= (b_{12} + 2b_{66})n_1/D_{11},\\
P_{17} &= 4(B_{26}n_1/R - D_{26}n_3)/D_{11}, & P_{18} &= -4D_{16}n_1/D_{11},\\
P_{19} &= (D_{261}n_3 - B_{261}n_1/R)/D_{11}, & P_{20} &= 3D_{161}n_1/D_{11},\\
P_{21} &= (D_{222}n_3 - B_{222}n_1/R + \bar{\rho}_1^{21}\omega^2n_1)/D_{11}, & P_{22} &= d_2'n_1/D_{11};
\end{aligned}
\tag{B5}
$$

4th Equation:

$$\begin{aligned} K_{29} &= -(B_{661}n_2 + \bar{\rho}_0^{11}\omega^2)/D_{1111}, & K_{30} &= -B_{111}/D_{1111}, \\ K_{31} &= b'_{261}n_2/D_{1111}, & K_{32} &= -b'_{161}/D_{1111}, \\ K_{33} &= -(B_{121}/R - d'_1 n_2 - \bar{\rho}_1^{11}\omega^2)/D_{1111}, & K_{34} &= D_{111}/D_{1111}, \\ K_{35} &= -(D_{661}n_2 - A_{5511} + \bar{\rho}_0^{12}\omega^2)/D_{1111}, & K_{36} &= -(D_{2612}n_2 - A_{4512})/D_{1111}, \\ K_{37} &= -D_{1612}/D_{1111}, & P_{23} &= -2B_{161}n_1/D_{1111}, \\ P_{24} &= -(b'_{121} + b'_{661})n_1/D_{1111}, & P_{25} &= -(B_{261}n_1/R - D_{261}n_3)/D_{1111}, \\ P_{26} &= 3D_{161}n_1/D_{1111}, & P_{27} &= -2D_{1611}n_1/D_{1111}, \\ P_{28} &= -d''_3 n_1/D_{1111}; \end{aligned} \tag{B6}$$

5th Equation:

$$\begin{aligned} K_{38} &= -B_{262}n_2/D_{6622}, & K_{39} &= -B_{162}/D_{6622}, \\ K_{40} &= -(b'_{222}n_2 + I_{23}\omega^2)/D_{6622}, & K_{41} &= -b'_{662}/D_{6622}, \\ K_{42} &= -(B_{262}/R - 3D_{262}n_2)/D_{6622}, & K_{43} &= D_{162}/D_{6622}, \\ K_{44} &= -(D_{2612}n_2 - A_{4512})/D_{6622}, & K_{45} &= -D_{1612}/D_{6622}, \\ K_{46} &= -(D_{2222}n_2 - A_{4422} + \bar{\rho}_0^{22}\omega^2)/D_{6622}, \\ P_{29} &= -b'_2 n_1/D_{6622}, & P_{30} &= 2b_{262}n_1/D_{6622}, \\ P_{31} &= -(B_{2222}n_1/R - D_{2222}n_3 - \bar{\rho}_1^{21}\omega^2 n_1)/D_{6622}, & P_{32} &= d'_2 n_1/D_{6622}, \\ P_{33} &= -d''_3 n_1/D_{6622}, & P_{34} &= -2D_{2622}n_1/D_{6622}, \end{aligned} \tag{B7}$$

where

$$\begin{aligned} n_1 &= n/R, & n_2 &= -(n/R)^2, \\ n_3 &= -(n/R)^3, & n_4 &= (n/R)^4, \end{aligned} \tag{B8}$$

and

$$\begin{aligned} I_{20} &= \rho_0 - n_2\rho_2, & I_{21} &= \rho_0 + (2\rho_1 + \rho_2/R)/R, \\ I_{22} &= \rho_1 - \rho_2/R, & I_{23} &= \bar{\rho}_0^{21} + \bar{\rho}_1^{21}/R. \end{aligned} \tag{B9}$$

References

1. A. W. Leissa, *Vibration of Shells*. NASA SP-266 (1973) 435 pp. Reprinted, by the Acoustical Society of America (1994).
2. A. K. Noor, Bibliography of monographs and surveys on shells. *Appl. Mech. Rev.* 43 (1990) 223–234.
3. M. S. Qatu, Review of shallow shell vibration research. *The Shock and Vibration Digest* 24 (1992) 3–15.

4. K. P. Soldatos, Review of three dimensional dynamic analysis of circular cylinders and cylindrical shells. *Appl. Mech. Rev.* 47 (1994) 501–516.
5. K. M. Liew, C. M. Lim and S. Kitipornchai, Vibration of shallow shells: A review with bibliography. *Appl. Mech. Rev.* 50 (1997) 431–444.
6. R. M. Jones, *Mechanics of Composite Materials*. New York: Hemisphere (1975) 365 pp.
7. A. A. Khdeir, J. N. Reddy and D. Frederick, A study of bending, vibration and buckling of cross-ply circular cylindrical shells with various shell theories. *Int. J. Eng. Sci*. 27 (1989) 1337–1351.
8. L. Librescu, A. A. Khdeir and D. Frederick, A shear deformable theory of laminated composite shallow shell-type panels and their response analysis I: Free vibration and buckling. *Acta Mech.* 76 (1989) 1–33.
9. A. A. Khdeir and J.N. Reddy, Influence of edge conditions on the modal characteristics of cross-ply laminated shells. *Comp. Struct.* 34 (1990) 817–826.
10. A. Nosier and J. N. Reddy, Vibration and stability analyses of cross-ply laminated circular cylindrical shells. *J. Sound Vibr.* 157 (1992) 139–159.
11. T. Timarci and K. P. Soldatos, Comparative dynamic studies for symmetric cross-ply circular cylindrical shells on the basis of a unified shear deformable shell theory. *J. Sound Vibr.* 187 (1995) 609–624.
12. T. Timarci, *Vibrations of Composite Laminated Cylindrical Shells*. PhD Thesis, Dept. Theo. Mech., Univ. Nottingham (1995).
13. C.-T. Sun and J. M. Whitney, Axisymmetric vibrations of laminated composite cylindrical shells. *J. Acoust. Soc. Am.* 55 (1974) 1238–1246.
14. K. P. Soldatos and J. Q. Ye, Axisymmetric static and dynamic analysis of laminated hollow cylinders of monoclinic elastic layers. *J. Sound Vibr.* 184 (1995) 245–259.
15. M. E. Vanderpool and C. W. Bert, Vibration of a materially monoclinic, thick-wall circular cylindrical shell. *A.I.A.A.J.* 19 (1981) 634–641.
16. K. P. Soldatos, A refined laminated plate and shell theory with applications. *J. Sound Vibr.* 144 (1991) 109–129.
17. Y. Narita, Y. Ohta, G. Yamada and Y. Kobayashi, Analytical method for vibration of angle-ply cylindrical shells having arbitrary edges. *A.I.A.A.J.* 30 (1992) 790–796.
18. W. Flügge, *Stresses in Shells*, 2nd edn. New York: Springer-Verlag (1973) 525 pp.
19. K. Forsberg, Influence of boundary conditions on the modal characteristics of thin elastic shells. *A.I.A.A.J.* 2 (1964) 2150–2157.
20. S. Cheng and B. P. C. Ho, Stability of heterogeneous aeolotropic shells. *A.I.A.A.J.* 1 (1963) 892–898.
21. C. W. Bert, J. L. Baker and D. M. Egle, Free vibrations of multilayered anisotropic cylindrical shells. *J. Comp. Mat.* 3 (1969) 480–499.
22. K. P. Soldatos and T. Timarci, A unified formulation of laminated composite, shear deformable, five-degrees-of-freedom cylindrical shell theories. *Comp. Struct.* 25 (1993) 165–171.
23. K. P. Soldatos, A transverse shear deformation theory for homogeneous monoclinic plate. *Acta Mech.* 94 (1992) 195–220.
24. J. Hadian and A. H. Nayfeh, Free vibration and buckling of shear deformable cross-ply laminated plates using the state-space concept. *Comp. Struct.* 48 (1993) 677–693.
25. A. A. Khdeir, A remark on the state-space concept applied to bending, buckling and free vibration of composite laminates. *Comp. Struct.* 59 (1996) 813–817.